PHYC 581: High Energy Astrophysics

Fall 2018

Lec 7: 09/12/2018

Fermi Acceleration (Cont'd):

We now turn to an alternative geometry in which the scattering of test particles occurs within a converging flow, arising across a shock. As we will see, the second-order process that we discussed last time will turn into a first-order one in this case.

First, let us discuss the shock waves in some detail.

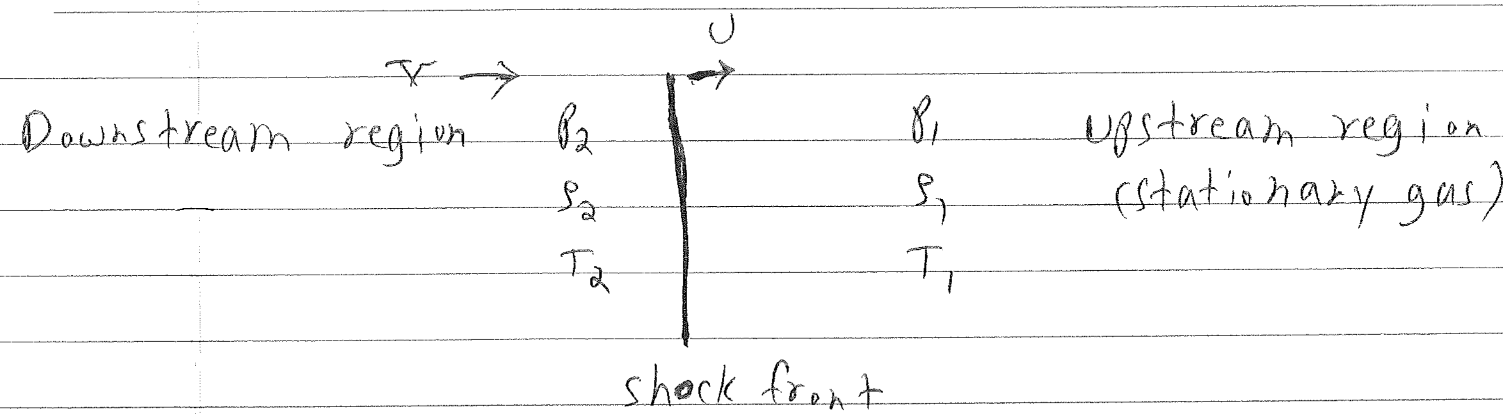
Shock waves are found ubiquitously in high energy astrophysics, and play a key role in many different astrophysical environments. It is a general property of perturbations in a gas that they are propagated away from their source at the speed of sound in the medium, c_s, given by:

$$c_s = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_{ad}} = \sqrt{\frac{\gamma P}{\rho}} \qquad \left(\gamma \equiv \frac{c_P}{c_V} = \frac{c_V + 1}{c_V}\right)$$

Here "ad" refers to the adiabatic condition.

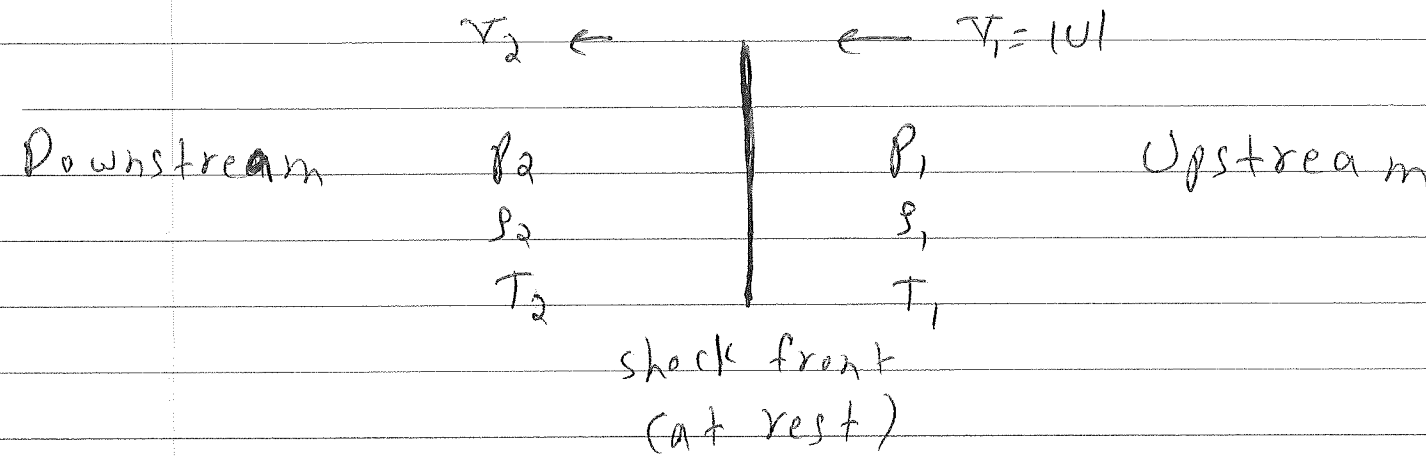
If the source and the gas have a relative velocity that is greater than c_s, then the disturbance cannot behave like a sound wave at all. There will be a discontinuity between the regions behind and ahead of the disturbance. These discontinuities are called shock waves.

Let us focus on a plane shock wave and assume abrupt discontinuity between the two regions of fluid flow:



It is convenient to transform to a reference frame moving at velocity U, in which the shock front is at stationary.

In this frame, the upstream region and the downstream region move at respective velocities v_1, v_2;



The behavior of the gas on passing through the shock front is described by a set of conservation relations. First, mass is conserved on passing through the discontinuity:

$$\rho_1 v_1 = \rho_2 v_2$$

Second, the energy flux is continuous. The energy flux through the shock front is $\rho_1 v_1 \left(\frac{1}{2} v_1^2 + w_1\right)$ and $\rho_2 v_2 \left(\frac{1}{2} v_2^2 + w_2\right)$ in the upstream and downstream regions respectively. Here, $w = PV + \epsilon$ is the enthalpy per unit mass, where ϵ is the internal energy per unit mass and $V = \rho^{-1}$ is the specific volume. Note that the $\frac{1}{2} v_1^2$ term is the kinetic energy per unit mass that arises because of the net flow of the gas. Conservation of energy flux therefore implies:

$$\rho_1 v_1 \left(\frac{1}{2} v_1^2 + w_1\right) = \rho_2 v_2 \left(\frac{1}{2} v_2^2 + w_2\right)$$

Finally, the momentum flux through the shock front must be continuous, which results in:

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2$$

We note that, as expected, the momentum flux is conserved if pressure is the same in the upstream and downstream regions $P_1 = P_2$.

For an ideal gas, we have $w = \frac{\gamma P V}{\gamma - 1} = \frac{\gamma}{\gamma - 1} \frac{P}{\rho}$. The mass flux is denoted by $j = \rho_1 v_1 = \rho_2 v_2$. We then find:

$$j^2 = \frac{P_2 - P_1}{V_1 - V_2} \qquad (V_1 = \rho_1^{-1},\ V_2 = \rho_2^{-1})$$

In addition:

$$v_1 - v_2 = j\,(V_1 - V_2) = \left[(P_2 - P_1)(V_1 - V_2)\right]^{\frac{1}{2}}$$

From the conservation of energy flux we obtain:

$$(w_1 - w_2) + \frac{1}{2}(V_1 + V_2)(P_2 - P_1) = 0$$

Using the relation $w = \frac{\gamma P V}{\gamma - 1}$ for an ideal gas, this leads to:

$$\frac{v_2}{v_1} = \frac{P_1(\gamma+1) + P_2(\gamma-1)}{P_1(\gamma-1) + P_2(\gamma+1)}$$

One can show that:

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1}$$

Here, M_1 is the Mach number in the upstream region:

$$M_1 \equiv \frac{v_1}{c_{s,1}}, \qquad c_{s,1} = \sqrt{\frac{\gamma P_1}{\rho_1}}$$

We then find:

$$\frac{v_1}{v_2} = \frac{\gamma+1}{(\gamma-1) + \frac{2}{M_1^2}}$$

For very strong shocks, $v_1 \gg c_{s,1}$, this results in:

$$\frac{v_1}{v_2} = \frac{\gamma+1}{\gamma-1}$$

For an ideal monatomic gas, we have $\gamma = \frac{5}{3}$, and hence:

$v_1 = 4v_2 \Rightarrow v_2 = \frac{|U|}{4}$

In the rest frame of the upstream region, gas in the downstream region moves at a speed $\frac{3U}{4}$ behind the shock front.

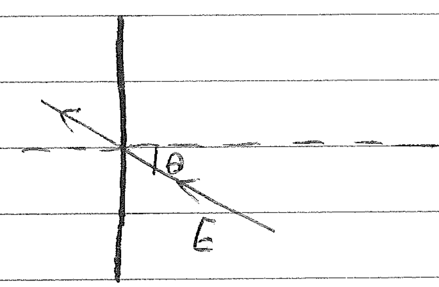
We now discuss the actual process of particle acceleration.

A particle crossing from the upstream to dowstream sides of the shock acquires an increase in its energy according to:

$E' = \gamma \left(E + \frac{3}{4} U p_x \right)$

$\gamma = \left[1 - \left(\frac{3U}{4c} \right)^2 \right]^{-\frac{1}{2}} \simeq 1$

$p_x = \frac{E}{c} \cos\theta$



shock front

Here, we have assumed that the particle that crosses the shock fround is relativistic, $E = \frac{p}{c}$, and the shock is non-relativistic $U \ll c$.

The energy increase is:

$$\Delta E = E' - E \Rightarrow \frac{\Delta E}{E} \simeq \frac{3}{4}\frac{U}{c}\cos\theta$$

Integrating over the incident angle $0 \leq \theta \leq \frac{\pi}{2}$, we find:

$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{3}{4}\frac{U}{c}\int_0^{\frac{\pi}{2}} P_{(\theta)} \cos\theta \, d\theta \qquad P_{(\theta)} = 2\sin\theta\cos\theta$$

Thus:

$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{1}{2}\frac{U}{c}$$

After crossing the shock front, the particle's velocity vector is randomized through elastic scatterings off the gas particles in the downstream region. The particle gains another fractional increase of $\frac{1}{2}\frac{U}{c}$ as it crosses the shock front back to the upstream region. This is the main difference with the simple one-dimensional example we considered earlier, in which the linear terms in head-on and catch-up collisions had opposite signs and cancelled out.

The average energy increase per round trip is:

$$\beta = 1 + \frac{1}{2}\frac{U}{c} + \frac{1}{2}\frac{U}{c} = 1 + \frac{U}{c}$$

Now, we need to work out the probability P that particle crosses the shock front back to the upstream region. According to kinetic theory, the flux of particles crossing a surface is $\frac{1}{4}nc$, where n is the number density of particles and particles are assumed to be relativistic. The flux represents the number of particles within one mean-free-path from the surface crossing it at incident angles $0 \le \theta \le \frac{\pi}{2}$.

In the case of the shock, however, the front moves at a velocity $\frac{1}{4}U$ relative to the downstream region. As a result, the flux to the downstream region is $\frac{1}{4}n(c + \frac{1}{2}U)$, while that from the downstream region is $\frac{1}{4}n(c - \frac{1}{2}U)$. This implies that the probability P for the particles to cross the shock front back is going to be:

$\rho = 1 - \frac{U}{c}$

Having found ρ and β, we can now derive the dependence of the spectrum on E. Starting with an initial energy E_0 and number N_0 in the upstream region, after k collisions we have:

$$E = E_0 \beta^k, \qquad N = N_0 \rho^k$$

Thus:

$$\ln\left(\frac{E}{E_0}\right) = k \ln\beta, \qquad \ln\left(\frac{N}{N_0}\right) = k \ln\rho$$

This results in:

$$\frac{N}{N_0} = \left(\frac{E}{E_0}\right)^{\frac{\ln\rho}{\ln\beta}}$$

since $\frac{U}{c} \ll 1$ (non-relativistic limit)

$$\frac{\ln\rho}{\ln\beta} = \frac{\ln(1-\frac{U}{c})}{\ln(1+\frac{U}{c})} \approx \frac{-\frac{U}{c}}{\frac{U}{c}} = -1$$

Therefore:

$$\frac{N}{N_0} \approx \left(\frac{E}{E_0}\right)^{-1} \Rightarrow \frac{dN}{dE} \equiv N(E) \propto E^{-2}$$

The value of the exponent is remarkably close to the "observed" universal value of -2.5.